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Scattering of Water Waves in an Ocean of Finite Depth having a Surface Discontinuity with an Ice-cover on One Half and Free Surface Subject to Surface Tension on the Other

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Abstract – The phenomenon of scattering of water waves is examined in presence of discontinuity at the free surface of an ocean of finite depth. The surface discontinuity is thought of as originating due to the presence of a semi-infinite free surface with surface tension on one half and a semi-infinite ice-covered ocean on the other half. Appropriate expressions for Green's functions are set up for the fluid occupied by the free surface with surface tension and for the fluid occupied by the ice-covered ocean. Employing Green's second integral theorem to the above mentioned Green's functions and the potential function the problem is reduced to finding out solutions of a pair of coupled Fredholm integral equation. Analytical expressions of reflection and transmission coefficients are obtained in integral forms.

Keywords – Fredholm integral equation, Green's function, Green's second identity, ice-covered ocean, reflection coefficient, surface discontinuity, surface tension, transmission coefficient.

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1 INTRODUCTION

The problems of scattering of water waves due to a discontinuity in the free surface are relevant to the ongoing research activities in the field of fluid dynamics, ocean engineering and coastal dynamics and have been dealt with by many authors. The free surface discontinuity arises because of two different types of boundary conditions on the two halves occupied by fluid. The difference in boundary conditions may arise, for instance, due to presence of two types of inertial surfaces of different surface densities of the two halves. Peters[1], Weitz and Keller[2] considered the Weiner-Hopf technique to study the propagation of surface waves at an inertial surface composed of a thin but uniform distribution of non-interacting floating materials, e.g. broken ice, floating mat, etc. on one side and the free surface on the other side. The problem of water wave scattering in presence of finite or semi-infinite elastic plate have been dealt with in [3,4,5,6,7]. Chung and Linton [8] found analytic expressions of hydrodynamic coefficients relating to scattering of water waves across a finite gap between two semi-infinite elastic plates using techniques of residue calculus. Chakraborti [9] formulated a singular integral equation approach to derive expressions for reflection and transmission coefficients for the problem concerning a semi-infinite inertial surface.

Another class of surface discontinuity arises in presence of a dock in the surface of the ocean. The free surface discontinuity arises, as an instance, when half the surface of water is free and the remaining half is covered by a dock extended upto infinity. The dock problem was formulated mathematically by Friedrich and Lewy [10]. Linton [11] made use of modified residue calculus technique to study the scattering problem in presence o a finite dock. Chakraborti, Mandal and Gayen [12] employed Fourier analysis and singular integral equation to examine the semi-infinite dock problem. Hermans [13] applied integral equations to study free-surface wave interaction with a thick flexible dock. Mandal and De [14] investigated surface wave propagation over small undulation at the bottom of the ocean with surface discontinuity using an eigen function expansion method. Gangopadhyay and Basu [15] studied scattering of capillary waves in front of a semi-infinite dock in an ocean with porous undulatory bottom using eigen function expansion and small perturbation technique.

Basu and Mandal [16] worked out diffraction of water waves by a deformation of the bottom in presence of surface tension in the free surface. P.F. Rhodes-Robinson

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[17] discussed fundamental singularities in the theory of scattering of water waves in presence of surface tension.

Meylan and Squire [18] considered scattering of water waves by finite ice-floes by reducing the problem to a Fredholm integral equation with logarithmic kernel. They constructed a Green's function for the boundary value problem of solving an ordinary differential equation subject to the boundary conditions on the ice-floes. Mandal and Basu [19] examined diffraction of water waves by a small cylindrical elevation of the bottom of a laterally unbounded ocean covered by an ice sheet using perturbation analysis. In this connection, Mandal and Maiti [20] investigated the problem of oblique wave scattering by cylindrical undulations on the bed of an ice-covered ocean by using a simplified perturbation analysis.

The present paper deals with the problem of scattering of water waves in an ocean of finite depth with a surface discontinuity having an ice-cover on one half and free surface with surface tension on the other. Appropriate expressions for Green's functions are constructed in two different zones. Next, Green's second identity is employed to the Green's functions and the potential function. The continuity of pressure and velocity at the junction along the vertical line of the surface of discontinuity is made use of.

Finally, a pair of coupled Fredholm integral equation is arrived at, which can be solved by standard numerical methods. Integrals representations of the hydrodynamic coefficients of interest such as reflection and transmission coefficients are arrived at.

2 MATHEMATICAL FORMULATION

A two-dimensional potential flow in an ocean of finite depth h is considered. A rectangular Cartesian co-ordinate system with y-axis vertically downwards along the depth of the ocean is chosen. The semi-infinite ice-covered surface of the ocean is assumed to occupy the semi-infinite region given by $0 \leq x < \infty,\ y = 0$ while the free surface subject to surface tension is assumed to occupy the semi-infinite region given by $-\infty < x < 0, y = 0$. The line of discontinuity is along $x = 0$. The undisturbed upper surface is considered to be along $y = 0$.

When waves traversing from $x = -\infty$ are incident on the line of discontinuity, the phenomenon of scattering takes place. Let $\psi(x,y) = Re\{\varphi(x,y)e^{-i\omega t}\}$ represent the velocity potential for the two-dimensional fluid region. Within the framework of linear theory and irrotational motion, the mathematical problem under consideration is to solve the boundary value problem in which the function φ satisfies the following Laplace equation along with certain boundary conditions:

$$\nabla^2\varphi = 0 \text{ in the entire fluid region} \qquad (1)$$

The free surface boundary condition subject to surface tension is given by:

$$K_1\varphi + \varphi_y + M\varphi_{yyy} = 0 \text{ on } y = 0, x < 0 \qquad (2)$$

where $K_1 = \frac{\omega^2}{g}$, ω is the angular frequency, g is the acceleration due to gravity; the surface tension parameter is M given by $M = T/\rho g$ where T is the coefficient of surface tension, ρ is the density of water. The ice-cover condition is given by:

$$K_2\varphi + \left(D\frac{\partial^4}{\partial x^4} + 1\right)\varphi_y = 0 \text{ on } y = 0, x > 0 \qquad (3)$$

where $D = Eh_0^3/12(1-\nu^2)\rho g$ is the ice-thickness parameter, E being the Young's modulus, ν is the Poisson's ratio of the material of the ice cover; the ice-cover being modeled as a thin sheet of an elastic plate of infinite extent having a very small thickness h_0 of which still a smaller part is immersed into water. The bottom boundary condition is given by:

$$\varphi_y = 0 \text{ on } y = h \qquad (4)$$

The far field behaviour of the potential function is described by:

$$\varphi \sim \begin{cases} \varphi^1(x,y) + R\varphi^1(-x,y) & as\ x \to -\infty \\ T\varphi^2(x,y) & as\ x \to \infty \end{cases} \qquad (5)$$

R and T respectively denote the reflection and the transmission coefficients of the present scattering problem.

$$\varphi^1(x,y) = \frac{coshk_{0,1}(h-y)}{coshk_{0,1}h}e^{ik_{0,1}x} \qquad (6)$$

$k_{0,1}$ being the unique positive real zero of $\Delta_1(k)$ where

$$\Delta_1(k) \equiv k(1 + Mk^3)sinhkh - K_1coshkh = 0 \qquad (7)$$

and $\varphi^2(x,y) = \frac{coshk_{0,2}(h-y)}{coshk_{0,2}h}e^{ik_{0,2}x}$ (8)

$k_{0,2}$ being the unique positive real zero of $\Delta_2(k)$ where

$$\Delta_2(k) \equiv k(1 + Dk^4)ksinhkh - K_2coshkh = 0 \qquad (9)$$

3 METHOD OF SOLUTION

Let $G_1(x,y;\xi,\eta)$ be the Green's function due a submerged line source at (ξ,η) for the region occupied by the free surface subject to surface tension. Let the corresponding function for the ice-covered region be $G_2(x,y;\xi,\eta)$. Then G_1 and G_2 satisfy the following conditions:

$\nabla^2 G_{1,2} = 0$ in the region

except at (ξ,η); $-\infty < x,\xi < \infty$ (10)

$$K_1G_1 + G_{1y} + MG_{1yyy} = 0 \text{ on } y = 0, x < 0 \qquad (11)$$

$$K_2G_2 + \left(D\frac{\partial^4}{\partial x^4} + 1\right)G_{2y} = 0 \text{ on } y = 0, x > 0 \qquad (12)$$

$G_{1,2} \to lnr$ where $r = \{(x-\xi)^2 + (y-\eta)^2\}^{\frac{1}{2}} \to 0$ (13)

$G_{1y} = G_{2y} = 0$ on $y = h$ (14)

The radiation conditions are given by:

$G_1 \sim$ multiple of $coshk_{0,1}(h-y)e^{ik_{0,1}|x-\xi|}$ as $|x-\xi| \to \infty$ (15)

$G_2 \sim$ multiple of $e^{ik_{0,2}|x-\xi|}$ as $|x-\xi| \to \infty$ (16)

The above boundary value problems for $G_j, j = 1,2$ have the following solutions as given by Thorne [21] :

$$G_j(x,y;\xi,\eta) = -ln\frac{r}{r'} - 2\int_0^\infty \frac{e^{-kh}sinhkysinhk\eta}{k}cosk(x-\xi)dk - 2\int_c \frac{\cosh(h-y)\,coshk(h-\eta)}{ksinhkh - K_j coshkh}\frac{cosk(x-\xi)}{coshkh}dk \quad (17)$$

where $r' = \{(x-\xi)^2 + (y+\eta)^2\}^{\frac{1}{2}}$. The path C is along the positive real axis in the complex k-plane. C is indented below the unique real positive roots $k_{0,1}$ and $k_{0,2}$ satisfying the transcendental equations (7) and (9) respectively.

Equation (17) has the following alternative representation:

$$G_j(x,y;\xi,\eta) = -4\pi i\int_0^\infty \frac{\mathrm{coshk_{0,j}(h-y)coshk_{0,j}(h-\eta)}}{2k_{0,j}h + sinh2k_{0,j}h}e^{ik_{0,j}|x-\xi|} - 4\pi\sum_{n=1}^\infty \frac{cosk_{n,j}(h-y)coshk_{0,j}(h-\eta)}{2k_{n,j}h + sin2k_{n,j}h}e^{-k_{n,j}|x-\xi|} \quad (18)$$

where $\pm ik_{n,1}$ and $\pm ik_{n,2} (n = 1,2, \dots)$ are the purely imaginary roots of the equations (7) and (9) respectively.

Green's function G_1 for the free surface having surface tension has the following explicit form as given by Rhodes-Robinson [22]:

$$G_1(x,y;\xi,\eta) = -4\pi i \cdot \frac{(1+Mk_{0,1}^2)coshk_{0,1}(h-y)cosk_{0,1}(h-\eta)e^{ik_{0,1}|x-\xi|}}{2k_{0,1}h(1+Mk_{0,1}^2) + (1+3Mk_{0,1}^2)sinh2k_{0,1}h} - 4\pi\sum_{n=1}^\infty \frac{(1-Mk_{n,1}^2)cosk_{n,1}(h-y)cosk_{n,1}(h-\eta)e^{-ik_{n,1}|x-\xi|}}{2k_{n,1}h(1-Mk_{n,1}^2) + (1-3Mk_{n,1}^2)sin2k_{n,1}} \quad (19)$$

Again, Green's function G_2 for the ice-covered half space has the following integral representation:

$$G_2(x,y;\xi,\eta) = -\int_\Gamma \frac{(1+Dk^4)kcoshk(h-y) - K_2 sinhk(h-y)}{k\Delta_2(k)}\mathrm{coshk}(h-\eta)\,e^{ik|x-\xi|}dk \quad (20)$$

where Γ is the path along the whole real axis with indentations above the pole at $k = -k_{0,2}$ and below the pole at $k = k_{0,2}$. The integral (20) can be evaluated by the method of residues, by forming a closed contour with Γ and a semi-circle of large radius above the real axis. The residues to be evaluated are at $k = k_{0,2}, \mu, -\bar{\mu}, ik_{n,2} (n = 1,2, \dots)$ and we finally get the following explicit expression for G_2:

$$G_2(x,y;\xi,\eta) = -4\pi\sum_{n=1}^\infty \frac{(1+Dk_{n,2}^4)cosk_{n,2}(h-y)cosk_{n,2}(h-\eta)}{2k_{n,2}h(1+Dk_{n,2}^4) + (1+5Dk_{n,2}^4)sin2k_{n,2}h}e^{-k_{n,2}|x-\xi|} - 4\pi i\frac{(1+Dk_{0,2}^4)coshk_{0,2}(h-y)coshk_{0,2}(h-\eta)}{2k_{0,2}h(1+Dk_{0,2}^4) + (1+5Dk_{0,2}^4)sinh2k_{0,2}h}e^{k_{0,2}|x-\xi|} - 4\pi i\frac{(1+D\mu^4)cosh\mu(h-y)cosh\mu(h-\eta)}{2\mu h(1+D\mu^4) + (1+5D\mu^4)sinh2\mu h}e^{i\mu|x-\xi|} - 4\pi i\frac{(1+D\bar{\mu}^4)cosh\bar{\mu}(h-y)cosh\bar{\mu}(h-\eta)}{2\bar{\mu}h(1+D\bar{\mu}^4) + (1+5D\bar{\mu}^4)sinh2\bar{\mu}h}e^{-i\bar{\mu}|x-\xi|} \quad (21)$$

Next, we make use of Green's integral theorem to the function $\varphi - e^{-iK_1x - K_1y}$ and $G_1(x \le 0, \xi \le 0)$ in the region bounded by the lines $y = 0, -X \le x \le 0; x = -X, 0 \le y \le h; y = h, -X \le x \le 0; x = 0, h \le y \le 0$; a small circle of radius ϵ and centre at (ξ,η). This leads to

$$2\pi\varphi(\xi,\eta) = 2\pi e^{-K_1\eta + iK_1\xi} + \int_0^\infty \left[\phi(0,y)\frac{\partial G_1}{\partial x}(0,y;\xi,\eta) - G_1(0,y;\xi,\eta)\frac{\partial\phi}{\partial x}(0,y)\right]dy + \chi(\xi,\eta);\ \xi \le 0 \quad (22)$$

where

$$\chi(\xi,\eta) = \int_0^\infty \left[iK_1e^{-K_1y}G_1(0,y;\xi,\eta) - e^{K_1y}\frac{\partial G_1}{\partial x}(0,y;\xi,\eta)\right]dy \quad (23)$$

To evaluate $\chi(\xi,\eta)$, we once again made use of Green's integral theorem to the functions $\Omega(x,y) = e^{-K_1y - iK_1x}$ and $G_1(x,y;\xi,\eta)$ in the region mentioned above. We are lead to:

$$\chi(\xi,\eta) = 2\pi\Omega(\xi,\eta) - 2\int_0^\infty e^{-K_1y}\frac{\partial G_1}{\partial x}(0,y;\xi,\eta)dy, \xi \le 0 \quad (24)$$

Using equation (24) in the equation (23) the ultimately making $X \to \infty$ and $\epsilon \to 0$ we obtain:

$$2\pi\varphi(\xi,\eta) = 2\pi e^{-K_1\eta}(e^{-iK\xi} + e^{iK\xi}) + \int_0^\infty \left[\varphi(0,y)\frac{\partial G_1}{\partial x}(0,y;\xi,\eta) - G_1(0,y;\xi,\eta)\frac{\partial\varphi}{\partial x}(0,y) - 2\pi e^{-K_1y}\frac{\partial G_1}{\partial x}(0,y;\xi,\eta)\right]dy, \xi \le 0 \quad (25)$$

Again we employ Green's integral theorem to the functions φ and $G_2(x \ge 0, \xi \ge 0)$ in the region bounded by the lines $y = 0, 0 \le x \le X; x = X, 0 \le y \le h; y = h, 0 \le x \le X; x = 0, 0 \le y \le h$; a small circle of radius ϵ and centre at

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(ξ, η). Making $X \to \infty$ and $\epsilon \to 0$, the following integral expression for $\varphi(\xi, \eta)$ is arrived at:

$$2\pi\varphi(\xi,\eta) = \int_0^\infty \left[G_2(0,y;\xi,\eta)\frac{\partial\varphi}{\partial x}(0,y) - \frac{\partial G_2}{\partial x}(0,y;\xi,\eta)\varphi(0,y)\right]dy, \xi \geq 0 \quad (26)$$

Now we consider the limiting values of the expressions (25) and (26) by letting $\xi \to 0-$ and $\xi \to 0+$ respectively. We get $\lim_{\xi\to 0+}\varphi(\xi,\eta)$ and $\lim_{\xi\to 0-}\varphi(\xi,\eta)$. Since φ is continuous along the line $x = 0$, taking the arithmetic mean of the two expressions, we have:

$$\varphi(0,\eta) = \frac{coshk_{0,1}(h-\eta)}{coshk_{0,1}h} + \frac{1}{2\pi}\int_0^h (G_2 - G_1)(0,y;0,\eta)\frac{\partial\varphi}{\partial x}(0,y)dy \quad (27)$$

Again differentiating the equations (25) and (26) with respect to ξ we make $\xi \to 0\mp$ in the resulting expressions. Using the continuity of $\frac{\partial\varphi}{\partial x}$ along the line $x = 0$ and taking the arithmetic mean of $\lim_{\xi\to 0-}\frac{\partial\varphi}{\partial\xi}(\xi,\eta)$ and $\lim_{\xi\to 0+}\frac{\partial\varphi}{\partial\xi}(\xi,\eta)$ we have:

$$\frac{\partial\varphi}{\partial\xi}(0,\eta) = i\frac{coshk_0(h-\eta)}{coshk_{0,1}h} + \frac{1}{2\pi}\int_0^h \frac{\partial^2}{\partial\xi\partial x}(G_1 - G_2)(0,y;0,\eta)\varphi(0,y)dy \quad (28)$$

The equations (27) and (28) are two coupled Fredholm integral equations with regular kernels for the unknown functions $\varphi(0,\eta)$ and $\frac{\partial\varphi}{\partial\xi}(0,\eta)$. This pair of integral equations may be solved by a suitable numerical method.

The hydrodynamic coefficients of interest in the present scattering problem are the reflection coefficient R and the transmission coefficient T which are obtained by letting $\xi \to -\infty$ in the equation (25) and $\xi \to \infty$ in (26) respectively and making use of G_j as given by (18):

$$R = \frac{2coshk_{0,1}h}{2k_{0,1}h + sinh2k_{0,1}h}\int_0^h coshk_{0,1}(h-y)\left[k_{0,1}\varphi(0,y) + i\frac{\partial\varphi}{\partial x}(0,y)\right]dy$$

and

$$T = \frac{2coshk_{0,2}h}{2k_{0,2}h + sinh2k_{0,2}h}\int_0^h coshk_{0,2}(h-y)\left[k_{0,2}\varphi(0,y) - i\frac{\partial\varphi}{\partial x}(0,y)\right]dy$$

6 CONCLUSION

The problem of scattering of water waves in an ocean of finite depth having a surface discontinuity with an ice-covered ocean on one half and free surface subject to surface tension on the other is explored by a novel method. After formulating the Green's functions for the two halves, one for the half occupied by the free surface with surface tension and the other for the half occupied by an ice-covered ocean, Green's second identity is made use of to the Green's functions and potential function thereby resulting in two coupled Fredholm integral equations the solutions of which may be computed by suitable numerical methods. The method is a simple one taking care of the continuity conditions of the potential function and its derivative along the vertical line of the surface of discontinuity. The analytical expressions for the reflection and transmission coefficients are arrived at. The present study is of practical use in its applications related to protecting coastal areas from the rough sea in the arctic regions.

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